

FACTORIZATION OF BIJECTIONS ONTO ORDERED SPACES

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ABSTRACT. We identify a class of subspaces of ordered spaces \mathcal{L} for which the following statement holds: If $f : X \rightarrow L \in \mathcal{L}$ is a continuous bijection of a zero-dimensional space X , then f can be re-routed via a zero-dimensional subspace of an ordered space that has weight not exceeding that of L .

1. INTRODUCTION

The paper studies factorization of continuous bijections onto subspaces of ordered spaces. In [4], Mardesic proved, in particular, that any continuous map of an n -dimensional compactum into a metric compact space can be re-routed via a metric compactum of dimension at most n . Deep research has been inspired by this result. We would like to mention a theorem of Pasynkov [5] that any continuous map of a Tychonoff Ind- n space into a metric space M admits a factorization with the middle space of Ind-dimension at most n and weight at most $w(M)$. This result implies, in particular, that if a space of Ind-dimension n admits a continuous injection into a metric space, then it admits a continuous injection into a metric space of Ind-dimension at most n . The paper is devoted to the following general problem:

Problem. Let a space X of dimension n admit a continuous bijection onto a space with property P . Does X admit a continuous bijection onto a space of dimension at most n and with property P ?

In this paper we are interested in inductive "ind" dimension. We first observe that for any $n > 2$ there exists a zero-dimensional space X that admits a continuous injection into \mathbb{R}^n but not into \mathbb{R}^{n-1} . We then observe that, if a zero-dimensional space continuously injects into \mathbb{R} , then it admits a continuous injections into the Cantor Set. The latter observation led the authors to the main result of this paper. To state the result, let \mathcal{L} be the class of all subspaces of ordered spaces that have a σ -disjoint π -base. Note that a subspace L of an ordered space is in \mathcal{L} if and only if L has a dense subset $\cup_n X_n$, where each X_n

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is a discrete (in itself) subspace. In our main result, Theorem 3.3, we prove that if $f : X \rightarrow L \in \mathcal{L}$ is a continuous bijection of a zero-dimensional space X , then f can be re-routed via a zero-dimensional subspace M of an ordered space of weight at most that of L . One may wonder if the class \mathcal{L} can be replaced by the class of all ordered spaces and their subspaces. The authors do not have an example to justify the restriction on the class \mathcal{L} but believe that there is a good chance for an example.

In notation and terminology, we will follow [1]. Spaces of ind-dimension 0 are called *zero-dimensional*. A linearly ordered space (abbreviated as LOTS) is one with the topology induced by some order. A subspace of a LOTS is a generalized ordered space (abbreviated as GO). It is a known and very useful fact that a Hausdorff space L is a GO-space, if its topology can be generated by a collection of convex sets with respect to some order on L (see, for example, [2]). A subset A of an ordered set $\langle S, \prec \rangle$ is \prec -convex if it is convex with respect to \prec . A collection \mathcal{U} of non-empty open sets of a space X is a π -base of X if for any non-empty open set O in X there exists $U \in \mathcal{U}$ such that $U \subset O$. All spaces are assumed Tychonoff. For the purpose of readability we will occasionally resort to an informal argument.

2. MOTIVATION

Ample research has been done to describe spaces that admit continuous injections into metric spaces. Assume that a space X has a certain dimension and admits a continuous injection into a metric space. It is natural to wonder if X admits a continuous injection into a metric space of the same dimension. As mentioned in the introduction, if dimension is in the sense of Ind or dim, then the affirmative answer is a simple corollary of the mentioned Pasynkov factorization theorem. In case of ind-dimension, additional analysis may be required. We start with examples.

Example 2.1. For each $N > 1$ there exists a separable space X of $\text{ind}(X) = 0$ that admits a continuous injection into \mathbb{R}^N but not into \mathbb{R}^{N-1} .

Construction. Put $D = \{\langle q_1, \dots, q_N \rangle : q_i \in \mathbb{Q}\}$ and $P = \mathbb{R}^N \setminus D$. The underlying set for our space is $\mathbb{R}^N = P \cup D$. We will define a new topology \mathcal{T} on \mathbb{R}^N so that $X = \langle \mathbb{R}^N, \mathcal{T} \rangle$ has desired properties. When \mathbb{R}^N is used with the Euclidean topology, we will refer to it by \mathbb{R}^N . In our new topology, points of D are declared isolated. Base neighborhoods at points of P will be defined in two stages.

Stage 1. For each map $f : D \rightarrow \mathbb{R}^{N-1}$, put

$$P_f = \{x \in P : f \text{ cannot be continuously extended to } D \cup \{x\}\},$$

where continuity at $x \in P$ is considered with respect to the Euclidean topology and D is regarded as discrete. Next let F be the set of all functions f from D to \mathbb{R}^{N-1} such that $|P_f| = 2^\omega$. For each $f \in F$, fix $x_f \in P_f$ so that $x_f \neq x_g$ for distinct $f, g \in F$. This can be done since $|F| = |P_f| = 2^\omega$ for each $f \in F$. Let us define a base neighborhood at x_f , for a fixed $f \in F$. Recall that f cannot be continuously extended to x_f with respect to the Euclidean topology at x_f . Therefore, there exists a sequence $\langle x_{f,n} \rangle_n$ of elements of D that converges to x_f in \mathbb{R}^N such that $\langle f(x_{f,n}) \rangle_n$ is not converging in \mathbb{R}^{N-1} . Sets in form $\{x_f\} \cup \{x_{f,i} : i > n\}$ will be base neighborhoods at x_f in our new topology. This completes Stage 1.

Stage 2. At the first stage we may not have defined base neighborhoods at all points of P . If $x \in P$ is such a point, fix an arbitrary sequence $\langle x_n \rangle_n$ of elements of D that converges to x in \mathbb{R}^N . A base neighborhood at x is in form $\{x\} \cup \{x_i : i > n\}$. This completes Stage 2 and definition of base neighborhoods at all points of X .

The topology \mathcal{T} is generated by the identified base neighborhoods. The space $X = \langle \mathbb{R}^N, \mathcal{T} \rangle$ has ind-dimension 0, which follows from the definition of base neighborhoods. The Euclidean topology of \mathbb{R}^N is a subtopology of X . Hence, X admits a continuous bijection onto \mathbb{R}^N . Let us show that X does not admit a continuous injection into \mathbb{R}^{N-1} . Fix any continuous function $h : X \rightarrow \mathbb{R}^{N-1}$. Assume that $f = h|_D$ is one-to-one. If P_f had the cardinality of continuum, then f would have been discontinuous at x_f due to Stage 1. Therefore, P_f has a smaller cardinality. Then there exists an irrational number r^* such that $L = \{\langle r^*, r_2, \dots, r_N \rangle : 0 \leq r_i \leq 1\} \subset P \setminus (P_f \cup D)$. By the definition of P_f , for every $x \in P \setminus P_f$, the function $h|_{D \cup \{x\}}$ is continuous at x with respect to the Euclidean topology. This fact and density of D in both X and \mathbb{R}^N imply the following claim.

Claim. $h|_L$ is continuous on L with respect to the Euclidean topology of L .

The set L with the Euclidean topology is homeomorphic to a closed disc of \mathbb{R}^{N-1} . Since $h|_L$ is one-to-one on L and L is compact with respect to the Euclidean topology, by Claim, $h(L)$ has non-empty interior in \mathbb{R}^{N-1} . Since D is dense in X , we conclude that $h(d) \in h(L)$ for some $d \in D$. Since $d \notin L$, we conclude that h is not one-to-one on X .

□

Example 2.1 may lead to an impression that with additional efforts one may construct a zero-dimensional space which is continuously injectable into \mathbb{R} but not into the Cantor Set. However, it is not the case as observed next.

Theorem 2.2. *Let f be a continuous injection of a zero-dimensional space X into \mathbb{R} . Then X can be continuously injected into the Cantor Set.*

Proof. Fix a countable dense subset A of $f(X)$. We may assume that $a, b \in A$ whenever a and b are immediate neighbors in $f(X)$. For each $a \in A$ and a positive integer n , fix a clopen neighborhood $U_{a,n}$ of $f^{-1}(a)$ such that $f(U_{a,n}) \subset (a - 1/n, a + 1/n)$. This is possible since X is zero-dimensional.

Define \mathcal{V} as follows. A set V is in \mathcal{V} if and only if it falls into one of the following three categories:

- (1) $V = U_{a,n}$ for some $a \in A$ and a positive integer n .
- (2) $V = f^{-1}((-\infty, a]) \setminus U_{a,n}$ for some $a \in A$ and a positive integer n .
- (3) $V = f^{-1}([a, \infty)) \setminus U_{a,n}$ for some $a \in A$ and a positive integer n .

The conclusion of the theorem follows from the following two claims.

Claim 1. Each element of \mathcal{V} is a clopen subset of X .

Proof of Claim 1. To prove Claim 1, it is enough to show that sets in category 2 are clopen. Each such set is closed as a difference of a closed set and an open set. Let us show open-ness. Since $f^{-1}(a)$ is a subset of $f^{-1}((-\infty, a]) \cap U_{a,n}$, we conclude that $f^{-1}((-\infty, a]) \setminus U_{a,n} = f^{-1}((-\infty, a)) \setminus U_{a,n}$. The right hand side of the equality is open as the difference of open and closed sets. The claim is proved.

Claim 2. \mathcal{V} is a countable point-separating family.

Proof of Claim 2. Countability follows from countability of A and the set of integers. Let x and y be two distinct points of X . Since f is one-to-one, we may assume that $f(x) < f(y)$. Recall that A is dense in $f(X)$ and contains any two points that are immediate neighbors in $f(X)$. Therefore, there exists $a \in A$ such that $f(x) \leq a < f(y)$ or $f(x) < a \leq f(y)$.

Assume that $f(x) < a < f(y)$. Let n be a positive integer such that $f(x) < a - 1/n$. Then x is an element of $f^{-1}((-\infty, a]) \setminus U_{a,n}$ and y is not.

Assume that $f(x) = a < f(y)$. Let n be a positive integer such that $a + 1/n < f(y)$. Then x is an element of $U_{a,n}$ and y is not. The case $f(x) < a = f(y)$ is treated similarly. \square

Note that the zero-dimensional topology constructed in the proof of the above theorem need not be comparable with the topology on X determined by the given map into \mathbb{R} . Let \mathcal{T} denote the topology generated by the union of the these two subtopologies. Clearly, $Y = \langle X, \mathcal{T} \rangle$ is a zero-dimensional separable metrizable space. Thus, the following factorization version of Theorem 2.2 holds.

Corollary 2.3. *Let X be zero-dimensional and let $f : X \rightarrow \mathbb{R}$ be a continuous injection. Then there exist a subspace Y of the Cantor Set and continuous maps $g : X \rightarrow Y$ and $h : Y \rightarrow \mathbb{R}$ such that $f = h \circ g$.*

It is natural to wonder if the target space \mathbb{R} , in statements 2.2 and 2.3, can be replaced by any (generalized) ordered space. This leads to our main result presented in the next section.

3. FACTORIZATION THEOREM

In this section, we generalize statements 2.2 and 2.3 for ranges from a sufficiently wide class. As demonstrated in the previous section, factorization version (Corollary 2.3) is an effortless corollary of injectability into the Cantor Set (Theorem 2.2) if the target space is the reals. In general, a topology generated by the union of two GO-topologies need not be a GO-topology (see [6] for examples). We, therefore, prove factorization version directly. We will generalize our statements to GO-spaces from the following class.

Definition of class \mathcal{L} . *A generalized ordered space L is in \mathcal{L} if and only if L has a dense subset which is a countable union of discrete (in themselves) subspaces.*

Another way to define this class is as follows.

Another Definition of class \mathcal{L} . *A generalized ordered space L is in \mathcal{L} if and only if L has a σ -disjoint π -base.*

One can easily show that all separable GO-spaces as well as all subspaces of any ordinal are in the class. The long line as well as the lexicographical product of any ordinal and $[0, 1)$ are in the class too. Note that the class need not contain subspaces of its members. In our proofs, we will be using the second definition of the class \mathcal{L} . For completeness, we next prove that the definitions are indeed equivalent.

Lemma 3.1. *A GO-space L has a σ -disjoint π -base if and only if L has a dense subspace that is a countable union of discrete subspaces.*

Proof. (\Rightarrow). Fix a σ -disjoint π -base and then select a point from each element of the base. The set of selected points is as desired.

(\Leftarrow) Let A_0 be the set of all isolated points of L and $M = L \setminus \overline{A_0}$. If M is empty, then $\{\{a\} : a \in A_0\}$ is a desired π -base. We now assume that M is not empty.

The set M has no isolated points. By Lemma's hypothesis, for any $n \geq 1$, we can find A_n , a discrete (in itself) subspace of M , so that $\cup_n A_n$ is dense in M .

Put $\mathcal{I}_0 = \{\{a\} : a \in A_0\}$. Next, for each $i, j > 0$, define \mathcal{I}_{ij} as follows: $I \in \mathcal{I}_{ij}$ if and only if I is a maximal non-empty convex subset of $M \setminus (A_i \cup A_j)$ that

is open in L . By maximality, \mathcal{I}_{ij} is a disjoint family of non-empty open convex subsets of L .

Let us show that $\mathcal{I}_0 \cup [\bigcup_{i,j} \mathcal{I}_{ij}]$ is a π -base for L . Fix any non-empty open subset U of L . We may assume that U is convex. If U has an isolated point a , then $a \in A_0$. We then have $\{a\} \in \mathcal{I}_0$ and $\{a\} \subset U$. Assume that U does not have isolated points. Since $\bigcup_n A_n$ is dense in M , we can find three distinct points a, b and c in $\bigcup_n A_n$ such that $a, c \in U$ and $a < b < c$. Fix indices i and j such that $a \in A_i$ and $c \in A_j$. Since both A_i and A_j are discrete in themselves and M has no isolated points, we conclude that $A_i \cup A_j$ is nowhere dense in U . Since $a < b < c$, the interval $(a, c)_L$ is a non-empty open subset of U that has no isolated points. Therefore, there exists $I \in \mathcal{I}_{ij}$ such that I is between a and c . Since U is convex, I is a subset of U . \square

The following folklore-type statement will be used in the main result and is proved for completeness.

Lemma 3.2. *Let R_n be an order relation on S for each n . Suppose that for any distinct $x, y \in S$ there exists n_{xy} such that either $xR_n y$ for all $n > n_{xy}$ or $yR_n x$ for all $n > n_{xy}$. Then R is a linear order on X , where*

$$R = \{\langle x, y \rangle : xR_n y \text{ for all } n > n_{xy}\}$$

Proof. Since every element of R is an element of R_n for some n , we conclude that $\langle x, x \rangle \notin R$ for any x . Therefore, R is irreflexive. To show comparability, fix any distinct $x, y \in S$. By Lemma's hypothesis, we may assume that $\langle x, y \rangle \in R_n$ for all $n > n_{xy}$. By the definition of R , $\langle x, y \rangle \in R$. To show transitivity, fix $\langle x, y \rangle, \langle y, z \rangle \in R$. Let $m = \max\{n_{xy}, n_{yz}\}$. Then, $\langle x, y \rangle, \langle y, z \rangle \in R_n$ for all $n > m$. By transitivity of R_n , we conclude that $\langle x, z \rangle \in R_n$ for all $n > m$. Therefore, $\langle x, z \rangle \in R$. \square

We are ready to prove our main result.

Theorem 3.3. *Let X be a zero-dimensional space and $L \in \mathcal{L}$. If $f : X \rightarrow L$ is a continuous bijection, then there exist a zero-dimensional GO-space M of weight at most that of L and continuous bijections $g : X \rightarrow M$ and $h : M \rightarrow L$ such that $f = h \circ g$.*

Proof. Fix a π -base $\bigcup_n \mathcal{I}_n$ of L , where each \mathcal{I}_n is a disjoint family of convex subsets of L . We may assume that $\mathcal{I}_0 = \emptyset$. We may also assume that the ground set for X and L is the same set S . Thus, $X = \langle S, \mathcal{T}_X \rangle$ and $L = \langle S, <_L, \mathcal{T}_L \rangle$. For simplicity we may write $<$ instead of $<_L$.

For each $n \in \omega$ we will define families to be later used in the construction of our middle space M .

Step 0. Put $\mathcal{A}_0 = \emptyset$, $\mathcal{B}_0 = <_L$, $<_0 = \mathcal{A}_0 \cup \mathcal{B}_0$, and $\mathcal{T}_0 = \mathcal{T}_L$.

Assumption. Assume that for each $k = 0, \dots, n-1$ we have defined relations \mathcal{A}_k and \mathcal{B}_k on S and a topology \mathcal{T}_k on S so that the following properties are satisfied:

- P1: $\mathcal{A}_m \subset \mathcal{A}_k$ whenever $m < k$.
- P2: $\mathcal{B}_m \supset \mathcal{B}_k$ whenever $m < k$.
- P3: $<_k = \mathcal{A}_k \cup \mathcal{B}_k$ is a linear order relation on S .
- P4: If $I \in \mathcal{I}_k$, then I contains a non-empty set which is clopen with respect to \mathcal{T}_k .
- P5: If U is $<_m$ -convex and clopen in $\langle S, \mathcal{T}_m \rangle$, then U is $<_k$ -convex and clopen in $\langle S, \mathcal{T}_k \rangle$ whenever $m < k$.
- P6: $\langle S, <_k, \mathcal{T}_k \rangle$ is a GO-space.
- P7: $\mathcal{T}_m \subset \mathcal{T}_k \subset \mathcal{T}_X$ whenever $m \leq k$.
- P8: The weight of $\langle S, \mathcal{T}_k \rangle$ is equal to the weight of L .

Note that the conditions are met for $k = 0$.

Step $n > 0$. Let \mathcal{C}_n be the collection of maximal subsets of $\langle S, <_{n-1}, \mathcal{T}_{n-1} \rangle$ that are open, connected, and without end-points. For each $C \in \mathcal{C}_n$ and $I \in \mathcal{I}_n$, such that $C \cap I \neq \emptyset$, select $y_{CI} \in C \cap I$. By P7, $C \cap I$ is open in X . Since X is zero-dimensional, there exists $V_{CI} \subset C \cap I$ such that $y_{CI} \in V_{CI}$ and V_{CI} is clopen in X .

Put

$$L_{CI} = \{x \in (C \cap I) \setminus V_{CI} : x <_{n-1} y_{CI}\} \text{ and } R_{CI} = \{x \in (C \cap I) \setminus V_{CI} : y_{CI} <_{n-1} x\}.$$

Define \mathcal{A}'_n as follows. A pair $\langle x, y \rangle$ is in \mathcal{A}'_n if and only if there exist $C \in \mathcal{C}_n$ and $I \in \mathcal{I}_n$ such that $I \cap C \neq \emptyset$ and one of the following holds:

- (1) $(x \in L_{CI}) \wedge (y \in V_{CI})$,
- (2) $(x \in L_{CI}) \wedge (y \in R_{CI})$,
- (3) $(x \in V_{CI}) \wedge (y \in R_{CI})$.

In words, when restricted to $I \cap C$, the relation \mathcal{A}'_n proclaims that L_{CI} is "less than" V_{CI} and that V_{CI} is "less than" R_{CI} .

Finally define \mathcal{A}_n , \mathcal{B}_n , and $<_n$ as follows.

$$\mathcal{A}_n = \mathcal{A}_{n-1} \cup \mathcal{A}'_n \text{ and } \mathcal{B}_n = \mathcal{B}_{n-1} \setminus \{\langle x, y \rangle : \langle x, y \rangle \text{ or } \langle y, x \rangle \text{ is in } \mathcal{A}'_n\},$$

$$<_n = \mathcal{A}_n \cup \mathcal{B}_n$$

Let us check P1-P3. Properties P1 and P2 follow from the definition.

Check of P3. We need to verify irreflexivity, comparability, and transitivity of $<_n$.

Irreflexivity: Fix $\langle x, y \rangle \in <_n$. Assume $\langle x, y \rangle \in \mathcal{B}_n$. By P2, $\langle x, y \rangle \in \mathcal{B}_{n-1}$. By P3, $x \neq y$. Assume $\langle x, y \rangle \in \mathcal{A}_{n-1}$. By P3, $x \neq y$. If $\langle x, y \rangle \notin \mathcal{A}_{n-1} \cup \mathcal{B}_n$, then $\langle x, y \rangle \in \mathcal{A}'_n$. Then x and y satisfy (1), (2), or (3) in the definition of \mathcal{A}'_n for some $C \in \mathcal{C}_n$ and $I \in \mathcal{I}_n$. Since the sets L_{CI}, V_{CI}, R_{CI} are disjoint, we conclude that $x \neq y$.

Comparability: Fix distinct x and y . Since $<_{n-1}$ is an order, we may assume that $\langle x, y \rangle \in <_{n-1}$. If $\langle x, y \rangle \in \mathcal{A}_{n-1}$, then, by P1, $\langle x, y \rangle \in \mathcal{A}_n \in <_n$. Otherwise, $\langle x, y \rangle \in \mathcal{B}_{n-1}$. If $\langle x, y \rangle \in \mathcal{B}_n$, then $\langle x, y \rangle \in <_n$. Otherwise, $\langle x, y \rangle \in \mathcal{B}_{n-1} \setminus \mathcal{B}_n$. By the definition of \mathcal{B}_n , we conclude that either $\langle x, y \rangle$ or $\langle y, x \rangle$ is in $\mathcal{A}'_n \in <_n$.

Transitivity: Fix $\langle x, y \rangle$ and $\langle y, z \rangle$ in $<_n$. We have four cases to consider.

Case $(\langle x, y \rangle, \langle y, z \rangle \notin \mathcal{A}'_n)$: The assumption implies that $\langle x, y \rangle, \langle y, z \rangle \in \mathcal{A}_{n-1} \cup \mathcal{B}_{n-1} = <_{n-1}$. Since $<_{n-1}$ is an order, $\langle x, z \rangle \in <_{n-1}$.

Assume first that $\langle x, z \rangle \notin \mathcal{A}'_n$. Then $\langle x, z \rangle \in \mathcal{B}_n \cup \mathcal{A}_{n-1} \subset <_n$.

Now assume that $\langle x, z \rangle \in \mathcal{A}'_n$. Then there exist $I \in \mathcal{I}_n$ and $C \in \mathcal{C}_n$ such that x, z are in $C \cap I$ and belong to distinct elements of $\mathcal{P} = \{L_{CI}, V_{CI}, R_{CI}\}$. Since C is connected with respect to \mathcal{T}_{n-1} and P7 holds for $n-1$, we conclude that $\mathcal{T}_0|_C = \mathcal{T}_{n-1}|_C$. Since I is convex with respect to $<_0$, the set $C \cap I$ is an open and connected subset of $\langle S, <_{n-1}, \mathcal{T}_{n-1} \rangle$. Since $x, z \in C \cap I$, $x <_{n-1} y$, and $y <_{n-1} z$, we conclude that $y \in C \cap I$. Then either x and y , or y and z are separated by \mathcal{P} . Therefore, either $\langle x, y \rangle \in \mathcal{A}'_n$ or $\langle y, z \rangle \in \mathcal{A}'_n$, contradicting the case's assumption. Therefore, the inclusion $\langle x, z \rangle \in \mathcal{A}'_n$ cannot occur.

Case $(\langle x, y \rangle \notin \mathcal{A}'_n, \langle y, z \rangle \in \mathcal{A}'_n)$: Since $\langle y, z \rangle \in \mathcal{A}'_n$, there exist $C \in \mathcal{C}_n$ and $I \in \mathcal{I}_n$ such that y and z are in $C \cap I$ and belong to distinct elements of $\{L_{CI}, V_{CI}, R_{CI}\}$. Due to similarity in reasoning, we may assume that $y \in L_{CI}$ and $z \in V_{CI}$.

Assume first that $x \in C \cap I$. Since $\langle x, y \rangle \notin \mathcal{A}'_n$, we conclude that $x \in L_{CI}$. Then $\langle x, z \rangle \in \mathcal{A}'_n \subset <_n$.

Assume now that $x \notin C \cap I$. Since $\langle x, y \rangle \notin \mathcal{A}'_n$, we conclude that $x <_{n-1} y$. Since $I \cap C$ is $<_{n-1}$ -connected, we conclude that $x <_{n-1} z$. Since $\{C \cap I : C \in \mathcal{C}_n, I \in \mathcal{I}_n, C \cap I \neq \emptyset\}$ is a disjoint family, neither $\langle x, z \rangle$ nor $\langle z, x \rangle$ is in \mathcal{A}'_n . Therefore, $\langle x, z \rangle \in \mathcal{A}_n \subset <_n$.

Case $(\langle x, y \rangle \in \mathcal{A}'_n, \langle y, z \rangle \notin \mathcal{A}'_n)$: Similar to the previous case.

Case $(\langle x, y \rangle, \langle y, z \rangle \in \mathcal{A}'_n)$: Since $\{C \cap I : C \in \mathcal{C}_n, I \in \mathcal{I}_n, C \cap I \neq \emptyset\}$ is a disjoint family, we conclude that x, y , and z belong to the same element of this family - $C \cap I$. Since $\langle y, z \rangle \in \mathcal{A}'_n$, we conclude that y cannot be in R_{CI} . Since $\langle x, y \rangle \in \mathcal{A}'_n$, we conclude that y cannot be in L_{CI} . Therefore, $x \in L_{CI}, y \in V_{CI}, z \in R_{CI}$. By the definition of \mathcal{A}'_n , we conclude that $\langle x, z \rangle \in \mathcal{A}'_n \subset <_n$.

Before verifying the remaining properties P4-P7 of $<_n$, let us make two claims for future reference.

Claim 1. $L_{CI} <_n V_{CI} <_n R_{CI}$.

The statement of this claim follows from our word description of \mathcal{A}'_n .

Claim 2. If $<_n$ differs from $<_{n-1}$ on $\{x, y\}$, then x is in one of L_{CI}, V_{CI}, R_{CI} and y is in one of the other two for some C and I .

To prove the claim, we may assume that $x <_{n-1} y$. If $\langle x, y \rangle$ were in \mathcal{A}_{n-1} , then $x <_n y$ would have been true by P1. Therefore, $\langle x, y \rangle \in \mathcal{B}_{n-1}$. Since $\langle x, y \rangle \notin <_n$, we conclude that $\langle x, y \rangle \notin \mathcal{B}_n$. By the definition of \mathcal{B}_n , we have $\langle y, x \rangle \in \mathcal{A}'_n$. The definition of \mathcal{A}'_n implies that x and y are in the described sets. The claim is proved.

For properties P4-P7, we define a new topology on S as follows:

\mathcal{T}_n is generated by $<_n$ -open sets, $<_{n-1}$ -convex sets clopen with respect to \mathcal{T}_{n-1} , and $\{V_{CI}, S \setminus V_{CI} : C \in \mathcal{C}_n, I \in \mathcal{I}_n, C \cap I \neq \emptyset\}$.

Let us verify P4-P7.

Check of P4. If I is not connected with respect to \mathcal{T}_{n-1} , then I contains an $<_{n-1}$ -convex set U which is clopen with respect to \mathcal{T}_{n-1} . By the definition of \mathcal{T}_n , the set U is in \mathcal{T}_n . Otherwise, I is a non-trivial connected set with respect to \mathcal{T}_{n-1} . Then there exists $C \in \mathcal{C}_n$ such that $I \cap C$ is not empty. Then $U = V_{CI}$ is as desired.

Check of P5. Assume that U is $<_m$ -convex and clopen in $\langle S, \mathcal{T}_m \rangle$ for some $m < n$. Then, by induction, U is $<_{n-1}$ -convex and clopen in $\langle S, \mathcal{T}_{n-1} \rangle$. By Claim 2, U is $<_n$ -convex. By the definition of \mathcal{T}_n , U is clopen in $\langle S, \mathcal{T}_n \rangle$.

Check of P6. Note that $\{V_{CI} : C \in \mathcal{C}_n, I \in \mathcal{I}_n, C \cap I \neq \emptyset\}$ consists of $<_n$ -convex sets. By Claim 2, every $<_{n-1}$ -convex set clopen with respect to \mathcal{T}_{n-1} is $<_n$ -convex. Therefore, \mathcal{T}_n is generated by $<_n$ -convex sets.

Check of P7. To show $\mathcal{T}_n \subset \mathcal{T}_X$, fix $U \in \mathcal{T}_n$. Note that all V_{CI} 's are open in X by construction. If U is $<_{n-1}$ -convex and clopen with respect to \mathcal{T}_{n-1} , then $U \in \mathcal{T}_X$ by P5 for $n-1$. Therefore, we may assume that $U = \{x : x <_n a\}$ for some $a \in S$. There are two cases.

Case 1: The assumption is that $a \in I \cap C$ for some $I \in \mathcal{I}_n$ and $C \in \mathcal{C}_n$. If $a \in L_{CI}$, then $\{x : x <_n a\} = \{x : x <_{n-1} a\} \setminus V_{CI}$. Since V_{CI} is clopen in X by construction and $\{x : x <_{n-1} a\}$ is open in X by assumption for $n-1$, the difference is open in X too. Case $a \in V_{CI}$ and $a \in R_{CI}$ are handled similarly.

Case 2: The assumption is that $a \notin I \cap C$ for any $I \in \mathcal{I}_n$ and $C \in \mathcal{C}_n$. By Claim 2, $\{x : x <_n a\} = \{x : x <_{n-1} a\}$. The right side is open in X by assumption.

To prove that $\mathcal{T}_{n-1} \subset \mathcal{T}_n$, fix $U \in \mathcal{T}_{n-1}$. We may assume that U is $<_{n-1}$ -convex. If U is clopen with respect to \mathcal{T}_{n-1} , then $U \in \mathcal{T}_n$ by definition. Therefore, we may assume that $U = \{x : x <_{n-1} a\}$ for some a . We have two cases.

Case 1: The assumption is that $a \in I \cap C$ for some $I \in \mathcal{I}_n$ and $C \in \mathcal{C}_n$. If $a \in L_{CI}$, then $\{x : x <_{n-1} a\} = \{x : x <_n a\} \cup \{x \in V_{CI} : x <_{n-1} a\}$. Since $<_n$ and $<_{n-1}$ coincide on V_{CI} , we conclude that $\{x \in V_{CI} : x <_{n-1} a\}$ is in \mathcal{T}_n . Hence, the union is in \mathcal{T}_n . Case $a \in V_{CI}$ and $a \in R_{CI}$ are handled similarly.

Case 2: The assumption is that $a \notin I \cap C$ for any $I \in \mathcal{I}_n$ and $C \in \mathcal{C}_n$. By Claim 2, $\{x : x <_{n-1} a\} = \{x : x <_n a\}$. The right side is in \mathcal{T}_n by definition.

Check of P8. Note that $\langle S, <_n, \mathcal{T}_n \rangle$ is obtained from $\langle S, <_{n-1}, \mathcal{T}_{n-1} \rangle$ by lifting V_{CI} and then inserting it between L_{CI} and R_{CI} for each $C \in \mathcal{C}_n$ and $I \in \mathcal{I}_n$ with non-empty intersection. Since both \mathcal{C}_n and \mathcal{I}_n are disjoint collections of open sets in $\langle S, <_{n-1}, \mathcal{T}_{n-1} \rangle$, we conclude that $\langle S, <_n, \mathcal{T}_n \rangle$ and $\langle S, <_{n-1}, \mathcal{T}_{n-1} \rangle$ have the same density, and hence weight. The latter has the same weight as L by P8 for $n - 1$.

The inductive construction is complete. Put $<_M = [\bigcup_n \mathcal{A}_n] \cup [\bigcap_n \mathcal{B}_n]$.

Claim 3. $<_M$ is an order on S .

By Lemma 3.2, to prove the claim it suffices to show that

$$<_M = \{\langle x, y \rangle : x <_n y \text{ for all large enough } n\}.$$

To prove the inclusion " \subset ", fix $\langle x, y \rangle \in <_M$. If $\langle x, y \rangle \in \mathcal{A}_N$ for some N , then $x <_n y$ for all $n > N$. If $\langle x, y \rangle \in \bigcap_n \mathcal{B}_n$, then $x <_n y$ for all $n > 1$. Therefore, $<_M \subset \{\langle x, y \rangle : x <_n y \text{ for all large enough } n\}$.

To prove the inclusion " \supset ", fix $\{x, y, K\}$ such that $x <_n y$ for all $n > K$. Assume that $\langle x, y \rangle \in \mathcal{A}_N$ for some N . Then $x <_M y$. Otherwise, $\langle x, y \rangle \in \mathcal{B}_n$ for all $n > K$. By P2, $\langle x, y \rangle \in \mathcal{B}_n$ for all $n \leq K$. Therefore, $\langle x, y \rangle \in \bigcap_n \mathcal{B}_n \subset <_M$. The claim is proved.

Let \mathcal{T}_M be the topology generated by $\bigcup_n \mathcal{T}_n$.

Claim 4. The weight of $\langle S, \mathcal{T}_M \rangle$ is equal to that of L .

The statement is a direct corollary of P8 and the definition of \mathcal{T}_M .

Claim 5. $\mathcal{T}_L \subset \mathcal{T}_M \subset \mathcal{T}_X$.

The statement of the claim follows from Property P7 and the definition of \mathcal{T}_M .

Claim 6. $\langle S, <_M, \mathcal{T}_M \rangle$ is a zero-dimensional GO-space.

To prove the claim, first observe that M is Hausdorff due to inclusion $\mathcal{T}_L \subset \mathcal{T}_M$. It is left to show that for any $U \in \mathcal{T}_M$ and $x \in U$, there exists a $<_M$ -convex set O that is clopen in M and $x \in O \subset U$.

Since \mathcal{T}_M is generated by $\bigcup_n \mathcal{T}_n$, we may assume that $U \in \mathcal{T}_n$ for some n . Let C_x be a maximal connected subset of U with respect to \mathcal{T}_n that contains x . We have the following cases:

Case 1: The assumption is $C_x = \{x\}$. Since $\langle S, <_n, \mathcal{T}_n \rangle$ is a GO-space, there exists a $<_n$ -convex set O which is a clopen with respect to \mathcal{T}_n and $x \in O \subset U$. Then, by P5, O is $<_k$ -convex and clopen with respect to $\langle S, <_k, \mathcal{T}_k \rangle$ for any $k \geq n$. Therefore, O is $<_M$ -convex and clopen in M .

Case 2: The assumption is that there exist a and b in C_x such that $a <_n x <_n b$. Since $\mathcal{T}_L \subset \mathcal{T}_n$ and $[a, b]_{<_n}$ is compact in \mathcal{T}_n , we conclude that $(a, x)_{<_n}$ and $(x, b)_{<_n}$ are non-empty open sets in L . Therefore, there exists $I \in \mathcal{T}_k$ for some k such that $I \subset (a, x)_{<_n}$. By P4 and P6, there exists a non-empty clopen set V_a in \mathcal{T}_k such that V_a is $<_k$ -convex and $V_a \subset I$. Fix $a' \in V_a$. Similarly find a $V_b \subset (x, b)_{<_n}$ that is nonempty, clopen, and convex in some $\langle S, <_m, \mathcal{T}_m \rangle$. Fix $b' \in V_b$. Put $n^* = \max\{k, m\}$. The set $(a', b')_{<_n} \setminus [V_a \cup V_b]$ contains x , and, by P5 and P7, is clopen with respect to \mathcal{T}_{n^*} . Since $\langle S, <_{n^*}, \mathcal{T}_{n^*} \rangle$ is a GO-space, there exists a clopen convex neighborhood O of x in $\langle S, <_{n^*}, \mathcal{T}_{n^*} \rangle$ that contains x and is contained in $(a', b')_{<_n} \subset U$. Therefore, O is convex and clopen in M . Hence, O is as desired.

Case 3: The assumption is that neither Case 1 nor Case 2 takes place. Then one of end-points of C_x is x . Then we proceed as in Case 2.

By Claims 3-6, the space $M = \langle S, <_M, \mathcal{T}_M \rangle$ and $h = g = id_S$ are as desired. \square

We would like to finish with a few questions that naturally arise from our discussion.

Question 3.4. *If L is linearly ordered in Theorem 3.3, can M be made linearly ordered too?*

Question 3.5. *Is there a GO-space outside of class \mathcal{L} for which the conclusion of the theorem fails.*

Question 3.6. *Let $f : X \rightarrow L \in \mathcal{L}$ be a continuous surjection and $Ind(X) = 0$. Does there exist a factorization for f with a zero-dimensional GO-space as a middle space?*

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